ON THE ASYMPTOTIC SIZES OF SUBSET ANDERSON–RUBIN AND LAGRANGE MULTIPLIER TESTS IN LINEAR INSTRUMENTAL VARIABLES REGRESSION

PATRIK GUGGENBERGER
University of California, San Diego, La Jolla, CA 92093-0508, U.S.A.

FRANK KLEIBERGEN
Brown University, Providence, RI 02912, U.S.A.

SOPHOCLES MAVROEIDIS
Oxford University, Oxford OX1 3UQ, United Kingdom

LINCHUN CHEN
University of California, San Diego, La Jolla, CA 92093-0534, U.S.A.

The copyright to this Article is held by the Econometric Society. It may be downloaded, printed and reproduced only for educational or research purposes, including use in course packs. No downloading or copying may be done for any commercial purpose without the explicit permission of the Econometric Society. For such commercial purposes contact the Office of the Econometric Society (contact information may be found at the website http://www.econometricsociety.org or in the back cover of Econometrica). This statement must be included on all copies of this Article that are made available electronically or in any other format.
ON THE ASYMPTOTIC SIZES OF SUBSET ANDERSON–RUBIN
AND LAGRANGE MULTIPLIER TESTS IN LINEAR
INSTRUMENTAL VARIABLES REGRESSION

BY PATRIK GUGGENBERGER, FRANK KLEIBERGEN,
SOPHOCLES MAVROEIDIS, AND LINCHUN CHEN

We consider tests of a simple null hypothesis on a subset of the coefficients of the exogenous and endogenous regressors in a single-equation linear instrumental variables regression model with potentially weak identification. Existing methods of subset inference (i) rely on the assumption that the parameters not under test are strongly identified, or (ii) are based on projection-type arguments. We show that, under homoskedasticity, the subset Anderson and Rubin (1949) test that replaces unknown parameters by limited information maximum likelihood estimates has correct asymptotic size without imposing additional identification assumptions, but that the corresponding subset Lagrange multiplier test is size distorted asymptotically.

KEYWORDS: Asymptotic size, linear IV model, size distortion, subset inference, weak instruments.

1. INTRODUCTION

IN THE LAST DECADE, we have witnessed an increase in the literature dealing with inference on the structural parameters in the linear instrumental variables (IVs) regression model. Its objective is to develop powerful tests whose asymptotic null rejection probability is controlled uniformly over a parameter space that allows for weak instruments. For a simple full vector hypothesis, satisfactory progress has been made and several robust procedures were introduced, most notably, the AR test by Anderson and Rubin (1949), the Lagrange multiplier (LM) test of Kleibergen (2002), and the conditional likelihood ratio (CLR) test of Moreira (2003).

An applied researcher is, however, typically not interested in simultaneous inference on all structural parameters, but in inference on a subset, like one component, of the structural parameter vector. Tests of a subset hypothesis are substantially more complicated than tests of a joint hypothesis since the unrestricted structural parameters enter the testing problem as additional nuisance parameters. Under the assumption that the unrestricted structural parameters are strongly identified, the above robust full vector procedures can...
be adapted by replacing the unrestricted structural parameters by consistently estimated counterparts; see Stock and Wright (2000), Kleibergen (2004, 2005), Guggenberger and Smith (2005), Otsu (2006), and Guggenberger, Ramalho, and Smith (2012), among others, for such adaptations of the AR, LM, and CLR tests to subset testing. Under the assumption of strong identification of the unrestricted structural parameters, the resulting subset tests were proven to be asymptotically robust with respect to the potential weakness of identification of the hypothesized structural parameters and, trivially, have non-worse power properties than projection-type tests. However, a long-standing question concerns the asymptotic size properties of these tests without any identification assumption imposed on the unrestricted structural parameters.

The current paper provides an answer to that question. We consider a linear IV regression model with a parameter space that does not restrict the reduced form coefficient matrix and thus allows for weak instruments. The parameter space imposes a Kronecker product structure on a certain covariance matrix, a restriction that is implied, for example, by conditional homoskedasticity. We study the asymptotic sizes of subset AR and LM tests when the unrestricted structural parameters are replaced by the limited information maximum likelihood (LIML) estimator. The null hypothesis allows for simultaneous tests on subsets of the slope parameters of the exogenous and endogenous regressors. As the main result of the paper, we prove that the subset AR test has correct asymptotic size. In contrast, we show that the asymptotic size of the subset LM test is distorted. We document this by deriving the asymptotic null rejection probability of the subset LM test under certain weak IV drifting parameter sequences. The probability can be substantially larger than the nominal size when the number of instruments is large. For example, for nominal size $\alpha = 5\%$ and two right hand side endogenous variables, we obtain asymptotic null rejection probabilities under certain parameter sequences of $9.6\%$, $15.5\%$, and $19.5\%$ when the number of instruments equals 10, 20, and 30, respectively. Given that the LM statistic appears as a main element in the subset CLR test, these findings indicate that the latter test also is asymptotically size distorted.

The paper is structured as follows. Section 2 introduces the model and discusses the asymptotic size properties of the subset AR test. Section 3 discusses the asymptotic size distortion of the subset LM test for the case with two endogenous regressors. An Appendix provides the proof of the main theoretical result and some additional technicalities.

We use the following notation. For a full column rank matrix $A$ with $n$ rows, let $P_A = A(A' A)^{-1} A'$ and $M_A = I_n - P_A$, where $I_n$ denotes the $n \times n$ identity matrix. If $A$ has zero columns, then we set $M_A = I_n$. The chi squared distribution with $k$ degrees of freedom and its $1 - \alpha$-quantile are written as $\chi^2_k$ and $\chi^2_{k, 1-\alpha}$. We write “wpa1” for “with probability approaching 1.”

are projected out. Typically, this leads to suboptimal power properties. In the linear IV model, a projected version of the AR test has been discussed in Dufour and Taamouti (2005). A refinement that improves on the power properties of the latter test was given in Chaudhuri and Zivot (2011).
2. ASYMPTOTIC SIZE OF THE SUBSET AR TEST

We consider the linear IV model

\[ y = Y\beta + W\gamma + \varepsilon, \]

\[ (Y : W) = Z(\Pi_Y : \Pi_W) + (V_Y : V_W), \]

where \( y \in \mathbb{R}^n \) and \( W \in \mathbb{R}^{n \times m_w} \) are endogenous variables, \( Y \in \mathbb{R}^{n \times m_Y} \) consists of endogenous and/or exogenous variables, \( Z \in \mathbb{R}^{n \times k} \) are instrumental variables, \( \varepsilon \in \mathbb{R}^n \), \( V_Y \in \mathbb{R}^{n \times m_Y} \) and \( V_W \in \mathbb{R}^{n \times m_w} \) are unobserved disturbances; \( V = [V_Y : V_W] \), and \( \beta \in \mathbb{R}^{m_Y}, \gamma \in \mathbb{R}^{m_w}, \Pi_Y \in \mathbb{R}^{k \times m_Y} \) and \( \Pi_W \in \mathbb{R}^{k \times m_w} \), with \( m = m_Y + m_w \), are unknown parameters, and \( k \geq m \). We are interested in testing the subset null hypothesis

\[ H_0: \beta = \beta_0 \quad \text{versus} \quad H_1: \beta \neq \beta_0. \]

This setup also covers general linear restrictions on the coefficients of the structural equation, since these can be expressed as (2) by appropriate reparameterization. Since the variables in \( Y \) can consist of endogenous or exogenous variables, we allow for simultaneous tests on elements of the slope parameters of the exogenous and endogenous regressors. For those variables in \( Y \) that are exogenous and are therefore part of the instrumental variables \( Z \), the disturbances in their first stage equation are all identical to zero.

To keep the exposition simple, we omit from the model stated in equation (1) any exogenous regressors whose coefficients remain unrestricted by the null hypothesis (2). When such exogenous regressors are present in the model, our results remain valid if we replace the variables that currently appear in the definition of the various statistics by the residuals that result from a regression of those variables on the included exogenous variables.\(^4\)

Denote by \( Z_i \) the \( i \)th row of \( Z \) written as a column vector, and analogously for other variables. We assume that the realizations \( (\varepsilon_i, V'_i, Z'_i)' \), \( i = 1, \ldots, n \), are i.i.d. with distribution \( F \). The distribution \( F \) may depend on \( n \), but for the most part we write \( F \) rather than \( F_n \) to simplify notation. Furthermore, \( E_F(Z_i(\varepsilon_i, V'_i)) = 0 \), where by \( E_F \) we denote expectation when the distribution of \( (\varepsilon_i, V'_i, Z'_i)' \) is \( F \). As made explicit below, we also assume homoskedasticity.

The Anderson–Rubin (AR) statistic (times \( k \)) (see Anderson and Rubin (1949)) for testing the joint hypothesis

\[ H^* : \beta = \beta_0, \gamma = \gamma_0 \]

\(^4\)In particular, suppose the structural equation is \( y = Y\beta + W\gamma + X\delta + \varepsilon \), where \( X \in \mathbb{R}^{n \times q} \) denotes the matrix of included exogenous regressors. Then, we need to replace \( (y : Y : W : Z) \) in the definitions (4), (5), (6), (10), and (17) by \( M_X(y : Y : W : Z) \).
is defined as $AR_n(\beta_0, \gamma_0)$, where

$$AR_n(\beta, \gamma) = \frac{1}{\hat{\sigma}_{\epsilon\epsilon}(\beta, \gamma)}(y - Y\beta - W\gamma)'P_Z(y - Y\beta - W\gamma),$$

$$\hat{\sigma}_{\epsilon\epsilon}(\beta, \gamma) = (1, -\beta', -\gamma')\hat{\Omega}(1, -\beta', -\gamma'),$$

and

$$\hat{\Omega} = \frac{1}{n-k}(y:Y:W)'M_Z(y:Y:W).$$

With slight abuse of notation, we define the subset AR statistic for testing $H_0$ as

$$AR_n(\beta_0) = \min_{\gamma \in \mathbb{R}^W} AR_n(\beta_0, \gamma).$$

For $\tilde{\gamma} = \arg \min_{\gamma} AR_n(\beta_0, \gamma)$, the subset AR statistic is then identical to

$$AR_n(\beta_0) = \frac{1}{\hat{\sigma}_{\epsilon\epsilon}(\beta_0, \tilde{\gamma})}(y - Y\beta_0 - W\tilde{\gamma})'P_Z(y - Y\beta_0 - W\tilde{\gamma}).$$

The joint AR statistic in (4) is a monotonic transformation of the concentrated log-likelihood of $(\beta, \gamma)$ under i.i.d. normal errors; see, for example, Hood and Koopmans (1953, p. 170) and Hausman (1983, p. 425). Minimizing the AR statistic with respect to $\gamma$ is therefore identical to maximizing the log-likelihood, so $\tilde{\gamma}$ is the constrained LIML estimator of $\gamma$ under the null hypothesis (2). The $k$-class formulation of the LIML estimator reads (see Hausman (1983))

$$\tilde{\gamma} = \left(W'\left(P_Z - \frac{\kappa_{\text{min}}}{n-k}M_Z\right)W\right)^{-1}W'\left(P_Z - \frac{\kappa_{\text{min}}}{n-k}M_Z\right)(y - Y\beta_0),$$

where $\kappa_{\text{min}}$ equals the smallest root of the characteristic polynomial:

$$|\kappa\hat{\Omega}_W - (y - Y\beta_0:W)'P_Z(y - Y\beta_0:W)| = 0,$$

with

$$\hat{\Omega}_W = \begin{pmatrix} 1 & 0 & 0 \\ -\beta_0 & 0 & 0 \\ 0 & \hat{\Omega} & 0 \end{pmatrix}.$$
If we substitute the $k$-class formulation of the LIML estimator (7) into the expression of the subset AR statistic (6), we obtain that the subset AR statistic equals the smallest root of the characteristic polynomial in (8):

$$\text{AR}_n(\beta_0) = \kappa_{\text{min}}.$$ (10)

It is well known (see, e.g., Stock and Wright (2000), Zivot, Startz, and Nelson (2006)) that when the unrestricted structural parameters $\gamma$ are strongly identified, $\text{AR}_n(\beta_0)$ has a $\chi^2_{k-mW}$ limiting distribution. This finding motivates the choice of the critical value for the subset AR test. The nominal size $\alpha$ subset AR test rejects the null in (2) if

$$\text{AR}_n(\beta_0) > \chi^2_{k-mW,1-\alpha}.$$ (11)

We next define the parameter space $\Lambda$ for $(\gamma, \Pi_W, \Pi_Y, F)$ under the null hypothesis in (2). For $U_i = (e_i, V_{W,i}')$,

$$\Lambda = \left\{ \lambda = (\gamma, \Pi_W, \Pi_Y, F) : \gamma \in \mathbb{R}^{mW}, \Pi_W \in \mathbb{R}^{k \times mW}, \Pi_Y \in \mathbb{R}^{k \times mY}, \right.$$ (12)

$$E_F(\|T_i\|^{2+\delta}) \leq M,$$

for $T_i \in \{Z_i, e_i, \text{vec}(Z_i, V_{W,i}), V_{W,i} e_i, e_i, V_{W,i}, Z_i\}$,

$$E_F(Z_i (e_i, V_i')) = 0,$$

$$E_F(\text{vec}(Z_i, U_i') (\text{vec}(Z_i, U_i'))') = (E_F(U_i U_i') \otimes E_F(Z_i Z_i')),$$

$$\lambda_{\text{min}}(A) \geq \delta$$ for $A \in \{E_F(Z_i Z_i'), E_F(U_i U_i')\}$,

for some $\delta > 0$ and $M < \infty$, where $\lambda_{\text{min}}(\cdot)$ denotes the smallest eigenvalue of a matrix, “$\otimes$” the Kronecker product of two matrices, and $\text{vec}(\cdot)$ the column vectorization of a matrix. The parameter space does not place any restrictions on the matrix $\Pi_W$ (or $\Pi_Y$) and thus allows for weak identification. Appropriate moment restrictions are imposed that allow for the application of Lyapunov central limit theorems (CLTs) and weak law of large numbers (WLLNs). As in Staiger and Stock (1997), it is assumed that the covariance matrix $E_F(\text{vec}(Z_i U_i') (\text{vec}(Z_i U_i'))'$ factors into the Kronecker product $(E_F(U_i U_i') \otimes E_F(Z_i Z_i'))$, which holds, for example, under conditional homoskedasticity. Note that $U_i = (e_i, V_{W,i}')$ does not include the reduced form error $V_{Y,i}$ for which no assumptions need to be imposed for the subset AR test. In particular, the setup allows for $V_{Y,i}$ to be identical to zero, which is the case when $Y$ is exogenous and an element of $Z$.

6Regarding the notation $(\gamma, \Pi_W, \Pi_Y, F)$ and elsewhere, note that we allow, as components of a vector, column vectors, matrices (of different dimensions), and distributions.
The asymptotic size of the subset AR test is defined as

\[
\text{AsySz}_{\text{AR}, \alpha} = \lim_{n \to \infty} \sup_{\lambda \in \Lambda} P_{\lambda}(\text{AR}_n(\beta_0) > \chi^2_{k-m_W,1-\alpha}),
\]

where \( P_{\lambda} \) denotes probability of an event when the null data generating process is pinned down by \( \lambda \in \Lambda \). The main result of the paper can now be formulated as follows.

**THEOREM 1:** Let \( 0 < \alpha < 1 \). Then the asymptotic size of the subset AR test equals \( \alpha \):

\[
\text{AsySz}_{\text{AR}, \alpha} = \alpha.
\]

By definition, the nominal size projected AR test (see, e.g., Dufour and Taamouti (2005)) rejects the null in (2) if the joint AR statistic \( \text{AR}_n(\beta_0, \gamma) \) in (4) exceeds \( \chi^2_{k,1-\alpha} \) for all \( \gamma \in \mathbb{R}^{m_W} \), that is, when \( \text{AR}_n(\beta_0) > \chi^2_{k,1-\alpha} \). Therefore, the nominal size \( \alpha \) subset AR and projected AR tests are based on the same test statistic, but the former test uses a strictly smaller critical value if \( m_W > 0 \). We therefore have the following corollary.

**COROLLARY 2:** Let \( m_W > 0 \). The nominal size \( \alpha \) projected AR test has asymptotic size strictly smaller than \( \alpha \). It is strictly less powerful than the nominal size \( \alpha \) subset AR test in (11).

**Comments:**
1. Theorem 1 and Corollary 2 combined imply that the subset AR test controls the asymptotic size and provides power improvements over the projected AR test.
2. Theorem 1 implies, in particular, that the limiting distribution of \( \text{AR}_n(\beta_0) \) under strong IV asymptotics provides a stochastic bound on its limiting distribution under weak IV asymptotics.
3. The results in Theorem 1 are specific to using the LIML estimator to estimate the unrestricted structural parameters. When we use another estimator to estimate them, Theorem 1 typically no longer holds, and the resulting subset AR test may be asymptotically size distorted. In particular, it can be shown that the subset AR test that is based on the two-stage least squares (2SLS) estimator of \( \gamma \) is asymptotically size distorted.
4. When \( m_Y = 0 \), \( \text{AR}_n(\beta_0) \) equals a version of the J statistic that is based on the LIML estimator; see, for example, Sargan (1958) and Hansen (1982). Theorem 1 implies that, asymptotically, the J statistic is bounded by a \( \chi^2_{k-m_W} \) distribution and that the resulting J test has correct asymptotic size irrespective of the degree of identification. Again, this robustness property does not hold if the J statistic is evaluated at the 2SLS rather than the LIML estimator.
5. The proof of Theorem 1 involves a number of steps. Some of these steps are discussed in Lemmas 3 and 4 in the Appendix. First, in Lemma 3, we construct an upper bound on the subset AR statistic. This upper bound is a finite
sample one, so it holds for every $n$. The conceptual idea behind the proof is that
if the asymptotic size of an $\alpha$-level test based on this upper bound statistic using
the $\chi^2_{k-m_W,1-\alpha}$ critical value is equal to $\alpha$, and the upper bound is sharp for some
drifting sequences of the parameter $\Pi_W$, then the asymptotic size of the subset
AR statistic is equal to $\alpha$ as well. We therefore proceed, in Lemma 4, by work-
ing out the asymptotic behavior of the upper bound of the subset Anderson–Rubin statistic. This upper bound equals a ratio, so we separately derive the
asymptotic behavior of the numerator and denominator. With respect to the
numerator, we show that its asymptotic behavior for a given drifting sequence
of $\Pi_W$ is $\chi^2_{k-m_W}$. For the denominator, we show that its asymptotic behavior is
such that it is greater than or equal to 1. Combining, we obtain that the upper
bound for a given drifting sequence of $\Pi_W$ is bounded by a $\chi^2_{k-m_W}$ random
variable. The next (main) technical hurdle that is addressed in the proof of
Theorem 1 is that this $\chi^2_{k-m_W}$ bound applies over all possible drifting sequences
of $\Pi_W$. The bound therefore even applies for drifting sequences that are such
that the asymptotic distribution of the subset AR statistic does not exist. The
asymptotic null rejection probability of the subset AR statistic along such se-
quences is, however, still controlled because the finite sample bound on the
subset AR statistic still applies and we have shown that its maximal rejection
frequency over all possible drifting sequences of $\Pi_W$ is controlled.

The proof strategy crucially hinges on the assumption of a Kronecker prod-
uct covariance matrix as specified in the parameter space $\Lambda$ in (12). We are
currently not able to drop this assumption and are not aware of any result in
the literature proving correct asymptotic size of plug-in type subset tests with-
out a Kronecker product assumption.

6. In linear IV, it is, for expository purposes, common to analyze the case
of fixed instruments, normal errors, and a known covariance matrix; see, for
example, Moreira (2003, 2009) and Andrews, Moreira, and Stock (2006). In
that case, the bound on the subset AR statistic simplifies as well:

\begin{align}
\text{AR}(\beta_0) & \leq \frac{z'_z M(\omega_W+z_{\nu_W})z_\varepsilon}{1 + \eta'[(\Theta_W + z_{\nu_W})(\Theta_W + z_\nu_W)]^{-1}\eta} \\
& \leq z'_z M(\omega_W+z_{\nu_W})z_\varepsilon \sim \chi^2_{k-m_W},
\end{align}

with $z_\varepsilon$ and $z_{\nu_W}$ independent standard normal, $k \times 1$ and $k \times m_W$, dimensional
random vectors/matrices; $\eta$ is a standard normal $m_W \times 1$ dimensional random
vector, and $\Theta_W = (Z'Z)^{1/2} \Pi_W \Sigma^{-1/2}_{WW,\varepsilon}$, with $\Sigma_{WW,\varepsilon} = \Sigma_{WW} - \sigma_{\varepsilon\varepsilon} \sigma_{\varepsilon\varepsilon}^{-1} \sigma_{\varepsilon\varepsilon}$, for $\Sigma = E(U_iU_i') = (\sigma_{\varepsilon\varepsilon} \sigma_{\varepsilon\varepsilon} \sigma_{\varepsilon\varepsilon} \Sigma_{WW})$. When $m_W = 1$ and the length of $\Theta_W$ goes to infinity,
the distribution of the subset AR statistic is $\chi^2_{k-m_W}$, which coincides with the
bound in (14).

7. To gain some further intuition for the result in Theorem 1, we note that
the subset AR statistic is identical to Anderson’s (1951) canonical correlation
statistic, which tests if a matrix is of reduced rank. A test of $H_0: \beta = \beta_0$ using
the subset AR statistic is therefore identical to a test of $H_0^*: \text{rank}(\Phi) = m_W$ using Anderson’s (1951) canonical correlation statistic in the model

$$ (y - Y\beta_0: W) = Z\Phi + (u: V_W), $$

with $u = \varepsilon + V_W\gamma$ and $\Phi \in \mathbb{R}^{k \times (m_W + 1)}$. The value for $\Phi$ implied under $H_0$ and (1) is

$$ \Phi = \Pi_W(\gamma: I_m), $$

which is a $k \times (m_W + 1)$ dimensional matrix of rank $m_W$.

The expression of the upper bound in the known covariance matrix case in (14) suggests that the distribution of the subset AR statistic is nondecreasing in the length of the normalized expression of $\Pi_W$, $\Theta_W$, when $m_W = 1$. The length of $\Theta_W$ reflects the strength of identification, so the distribution of the subset AR statistic is nondecreasing in the strength of identification. This property can be understood using the analogy with the statistic testing the rank of $\Phi$ discussed above. When the length of $\Theta_W$ is large, the smallest value of the rank statistic is attained at the reduced rank structure of $\Phi$ shown in (16). When the length of $\Theta_W$ is small, the smallest value of the rank statistic can be attained at a reduced rank value of $\Phi$ which results from a reduced rank structure in $\Pi_W$. This implies that this value of the rank statistic is less than the value attained at the reduced rank structure corresponding to (16). In the latter case, the rank statistic has a $\chi^2(k - m_W)$ distribution, so for small values of the length of $\Theta_W$, the distribution of the rank statistic is dominated by the $\chi^2(k - m_W)$ distribution.

3. SIZE DISTORTION OF THE SUBSET LM TEST

The joint AR test is known to have relatively poor power properties when the degree of overidentification is large. Recently, other tests were introduced that improve on the power properties, in particular, the LM test (Kleibergen (2002)) and the CLR test (Moreira (2003)). The purpose of this section is to show that the subset version of the LM test (Kleibergen (2004)) suffers from asymptotic size distortion. Because the LM statistic is an integral part of the CLR statistic, the subset CLR test quite certainly also suffers from asymptotic size distortion. Therefore, given the results in this section, if one attempts to improve further on the power properties of the subset AR test, the subset LM and CLR tests offer no easy solution.

To document the asymptotic size distortion, it is enough to show asymptotic overrejection of the null hypothesis under certain parameter sequences $\lambda_n = (\gamma_n, \Pi_{W,n}, \Pi_{Y,n}, F_n)$. Overrejection of the null of the subset LM test is pervasive under weak IV sequences, and we focus on just one particular choice below.
For simplicity, we consider only the case where \( m_Y = m_W = 1 \), that is, (2) tests a hypothesis on the scalar coefficient of the endogenous variable \( Y \). In that case, the subset LM test statistic is given by

\[
LM_n(\beta_0) = \frac{1}{\hat{\sigma}_{ee}(\beta_0, \tilde{\gamma})} (y - Y\beta_0 - W\tilde{\gamma})' P_{Z\Pi(\beta_0)} (y - Y\beta_0 - W\tilde{\gamma}),
\]

where

\[
\tilde{\Pi}(\beta_0) = (Z'Z)^{-1} Z' \left[(Y:W) - (y - Y\beta_0 - W\tilde{\gamma}) \frac{1}{\hat{\sigma}_{ee}(\beta_0, \tilde{\gamma})} \left(\begin{array}{c} -\beta_0 \\ -\tilde{\gamma} \end{array}\right) \hat{\Omega}(I_m)\right].
\]

When \( m_Y = m_W = 1 \), the nominal size \( \alpha \) subset LM test rejects the null in (2) if

\[
LM_n(\beta_0) > \chi^2_{1,1-\alpha}.
\]

The parameter space \( \Lambda \) is defined in this section as in (12), with \( U_i \) replaced by \( (\varepsilon_i, V_i') \) and with the additional restrictions \( E_F(\|T_i\|^2 + \delta) \leq M, \) for \( T_i \in \{Z_iV_i, \varepsilon_iV_i, V_i'\} \). These restrictions are needed for the subset LM test for the application of WLLNs and CLTs when constructing its limiting distribution.

To document asymptotic overrejection of the test in (19), we focus on parameter sequences \( \lambda_n = (\gamma_n, \Pi_{Y,n}, \Pi_{Y,n}, F_n) \) that are such that

\[
n^{1/2} Q^{1/2} \Pi_{Y,n}/\sqrt{\sigma_{YY}} \rightarrow h_{11} \in \mathbb{R}^k, \quad n^{1/2} Q^{1/2} \Pi_{W,n}/\sqrt{\sigma_{WW}} \rightarrow h_{12} \in \mathbb{R}^k,
\]

\[
\left(\frac{E_{F_n}(V_{Y,i})}{\sqrt{\sigma_{ee}\sigma_{YY}}}, \frac{E_{F_n}(V_{W,i})}{\sqrt{\sigma_{ee}\sigma_{WW}}}, \frac{E_{F_n}(V_{Y,i}V_{W,i})}{\sqrt{\sigma_{WW}\sigma_{YY}}}\right)' \rightarrow h_2 \in [-1, 1]^3,
\]

where \( Q = E_{F_n}(Z_iZ_i'), \sigma_{YY} = E_{F_n}(V_{Y,i}^2), \) and \( \sigma_{WW} = E_{F_n}(V_{W,i}^2) \). We denote such sequences \( \lambda_n \) by \( \lambda_{n,h} \), where \( h = (h_{11}, h_{12}, h_2)' \). The Appendix derives the limiting distribution \( LM_n(\beta_0) \) of \( LM_n(\beta_0) \) under \( \lambda_{n,h} \), see (59), when IVs are weak, that is, \( \|h_{11}\| < \infty \) and \( \|h_{12}\| < \infty \). The limiting distribution only depends on the parameters \( h_{1} = (h_{11}', h_{12}')' \) and \( h_2 \). In fact, it only depends on \( h_1 \) through \( \|h_{11}\|, \|h_{12}\|, h_{11}'h_{12} \). For example, when \( k = 5, 10, 15, 20, 25, \) and \( 30, \) then, under \( \lambda_{n,h} \) with, for example, \( \|h_{11}\| = 100, \|h_{12}\| = 1, h_{11}'h_{12} = 95, \)

\[7\text{We do not index } Q, \sigma_{YY}, \text{ etc. by } F_n \text{ or } n \text{ to simplify notation. Likewise for other expressions below, for example, } \Sigma, \sigma_{ee}, \text{ etc.}\]
$h_{21} = 0$, $h_{22} = .95$, and $h_{23} = .3$, the asymptotic null rejection probability is 5.7, 9.6, 12.9, 15.5, 17.7, 19.5%, respectively, for nominal size $\alpha = 5\%$. These probabilities are obtained by simulation using 500,000 simulation repetitions. They provide a lower bound for the asymptotic size of the subset LM test. The test is therefore size distorted and the distortion can be substantial when the number of instruments $k$ is large.

**APPENDIX**

The appendix provides the proof of Theorem 1 and the derivation of the limiting distribution of the subset LM statistic.

We first state two lemmas that are helpful to prove Theorem 1. Their proofs are given after the proof of Theorem 1 below.

**LEMMA 3:** Under the $H_0$ in (2), we have, wpa1,

\[
\text{AR}_n(\beta_0) = \min_{d \in \mathbb{R}^{1+m_W}} \frac{d' (\Sigma^{1/2} \hat{\Sigma}^{-1/2})' N_n' L_n N_n (\Sigma^{1/2} \hat{\Sigma}^{-1/2}) d}{d'd}
\]

and

\[
\text{AR}_n(\beta_0) \leq \frac{z_{e,n} M_{\tau n} z_{e,n}}{\xi_n},
\]

where

\[
\hat{\Sigma} = \begin{pmatrix}
\hat{\sigma}_{ee} & \hat{\sigma}_{eW} \\
\hat{\sigma}_{eW} & \hat{\Sigma}_{WW}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-\gamma & I_{m_W}
\end{pmatrix} \hat{\Omega}_W \begin{pmatrix}
1 & 0 \\
-\gamma & I_{m_W}
\end{pmatrix},
\]

\[
\Sigma = EF(U_i U_i') = \begin{pmatrix}
\sigma_{ee} & \sigma_{eW} \\
\sigma_{eW} & \Sigma_{WW}
\end{pmatrix},
\]

\[
\Sigma_{WW,e} = \Sigma_{WW} - \sigma_{eW} \sigma_{ee}^{-1} \sigma_{W,e}^{-1},
\]

and

\[
z_{e,n} = (Z' Z)^{-1/2} Z' \epsilon \sigma_{ee}^{-1/2} \in \mathbb{R}^k,
\]

\[
z_{V_W,n} = (Z' Z)^{-1/2} Z'(V_{W} - \epsilon \sigma_{ee}^{-1} \sigma_{eW}) \Sigma_{WW,e}^{-1/2} \in \mathbb{R}^{k \times m_W},
\]

\[
\Theta_n = (Z' Z)^{1/2} \Pi_w \Sigma_{WW,e}^{-1/2} \in \mathbb{R}^{k \times m_W},
\]

and

\[
\tau_n = \Theta_n + z_{V_W,n} \in \mathbb{R}^{k \times m_W},
\]

\[
\eta_n = (\tau_n' \tau_n)^{-1/2} \tau_n' z_{e,n} \in \mathbb{R}^{m_W},
\]

\[
\xi_n = (1, -\eta_n (\tau_n' \tau_n)^{-1/2}) (\Sigma^{-1/2} \hat{\Sigma} \Sigma^{-1/2}) (1, -\eta_n (\tau_n' \tau_n)^{-1/2})',
\]
and
\[
N_n = \begin{pmatrix}
(\tau_n' \tau_n)^{-1/2} \eta_n & 0 \\
0 & I_{m_W}
\end{pmatrix},
\]
\[
L_n = \begin{pmatrix}
z_{e,n}' M_{e,n} & 0 \\
0 & \tau_n'
\end{pmatrix}.
\]

The next lemma derives limiting expressions for \(\xi_n\) and \(z_{e,n}' M_{e,n} z_{e,n}\) under sequences \(\lambda_n = (\gamma_n, \Pi_{W,n}, \Pi_{Y,n}, F_n)\) of null data generating processes in \(\Lambda\) such that the factors of a singular value decomposition of
\[
(26) \quad \Theta(n) = Q^{1/2} n^{1/2} \Pi_{W,n} \Sigma_{W,W,e}^{-1/2} \in \mathcal{R}^{k \times m_W}
\]
converge, where again \(Q = E_{F_n}(Z_i Z_i')\). More precisely, by the singular value decomposition theorem (see, e.g., Golub and Van Loan (1989)), \(\Theta(n)\) can be decomposed into a product
\[
(27) \quad \Theta(n) = G_n D_n R_n',
\]
where \(G_n\) and \(R_n\) are \(k \times k\) and \(m_W \times m_W\) dimensional real orthonormal matrices, respectively, and \(D_n\) is a \(k \times m_W\) dimensional rectangular real diagonal matrix with nonnegative elements. The latter matrix is unique up to ordering of the diagonal elements. Let \(\mathcal{R}_\infty = \mathcal{R} \cup \{+\infty\}\).

**Lemma 4:** Let \(\lambda_n = (\gamma_n, \Pi_{W,n}, \Pi_{Y,n}, F_n)\) be a sequence of null data generating processes in \(\Lambda\) and \(\omega_n\) a subsequence of \(n\), and \(G_{\omega_n} D_{\omega_n} R_{\omega_n}'\) a singular value decomposition of \(\Theta(\omega_n)\). Assume \(G_{\omega_n} \to G\) and \(R_{\omega_n} \to R\) for orthonormal \(k \times k\) and \(m_W \times m_W\) dimensional matrices \(G\) and \(R\), respectively, and \(D_{\omega_n} \to D\) for a rectangular diagonal matrix \(D \in \mathcal{R}_\infty^{k \times m_W}\). Then, under \(\lambda_n\), we have
(i) \(\xi_{\omega_n} - (1 + p_{\omega_n}) = o_p(1)\) for some sequence of random variables \(p_{\omega_n}\) that satisfy \(p_{\omega_n} \geq 0\) with probability 1, and (ii) \(z_{e,\omega_n}' M_{\omega_n} z_{e,\omega_n} \to_d \chi^2_{k-m_W}\).

**Proof of Theorem 1:** By Lemma 3, we have, wpa1,
\[
(28) \quad \text{AR}_n(\beta_0) \leq \frac{z_{e,n}' M z_{e,n}}{\xi_n}.
\]

There exists a “worst case sequence” \(\lambda_n = (\gamma_n, \Pi_{W,n}, \Pi_{Y,n}, F_n) \in \Lambda\) of null data generating processes such that
\[
(29) \quad \text{AsySz}_{\text{AR},a} = \lim_{n \to \infty} \sup_{\lambda \in \Lambda} \sup_{n} P_{\lambda}(\text{AR}_n(\beta_0) > \chi^2_{k-m_W,1-a})
\]
\[
= \lim_{n \to \infty} P_{\lambda_n}(\text{AR}_n(\beta_0) > \chi^2_{k-m_W,1-a})
\]
\[
\leq \lim_{n \to \infty} P_{\lambda_n} \left( \frac{z_{e,n}' M z_{e,n}}{\xi_n} > \chi^2_{k-m_W,1-a} \right),
\]
where the first equality in (29) holds by definition of $\text{AsySz}_{\text{AR}, \alpha}$ in (13), the second equality by the choice of the sequence $\lambda_n$, $n \geq 1$, and the inequality holds by (28). Furthermore, one can always find a subsequence $\omega_n$ of $n$ such that, along $\lambda_{\omega_n}$, we have $G_{\omega_n} \rightarrow G$ and $R_{\omega_n} \rightarrow R$ for orthonormal matrices $G \in \mathbb{R}^{k \times k}$ and $R \in \mathbb{R}^{m_W \times m_W}$, $D_{\omega_n} \rightarrow D$ for a diagonal matrix $D \in \mathbb{R}^{k \times m_W}$, and

\[
\limsup_{n \to \infty} P_{\lambda_n} \left( \frac{z'_{e,n} M_{\tau_n} z_{e,n}}{\xi_n} > \chi^2_{k-m_W, 1-\alpha} \right) = \limsup_{n \to \infty} P_{\lambda_{\omega_n}} \left( \frac{z'_{e,\omega_n} M_{\tau_{\omega_n}} z_{e,\omega_n}}{\xi_{\omega_n}} > \chi^2_{k-m_W, 1-\alpha} \right),
\]

where $G_{\omega_n} D_{\omega_n} R'_{\omega_n}$ is a singular value decomposition of $\Theta(\omega_n)$.

But, under any sequence of null data generating processes $\lambda_n = (\gamma_n, \Pi_{W,n}, \Pi_{Y,n}, F_n)$ in $\Lambda$ and under any subsequence $\omega_n$ of $n$ such that $D_{\omega_n} \rightarrow D$, $G_{\omega_n} \rightarrow G$, and $R_{\omega_n} \rightarrow R$ under $\lambda_n$, we have, by Lemma 4(i) and (ii),

\[
\frac{z'_{e,\omega_n} M_{\tau_{\omega_n}} z_{e,\omega_n}}{\xi_{\omega_n}} \leq \frac{z'_{e,\omega_n} M_{\tau_{\omega_n}} z_{e,\omega_n} + o_p(1)}{\xi_{\omega_n}} \rightarrow_d \chi^2_{k-m_W}.
\]

This together with (29) and (30) shows that $\text{AsySz}_{\text{AR}, \alpha} \leq \alpha$. Under strong IV sequences, the asymptotic null rejection probability of the subset AR test equals $\alpha$; see Stock and Wright (2000). Thus, $\text{AsySz}_{\text{AR}, \alpha} = \alpha$. Q.E.D.

**Proof of Lemma 3:** The subset AR statistic $\text{AR}_n(\beta_0)$ equals the smallest root of the characteristic polynomial (8). From (1), we have that

\[
P_Z(\gamma - Y\beta_0 : W) = P_Z \left[ Z \Pi_W (\gamma : I_{m_W}) + (e : V_W) \begin{pmatrix} 1 & 0 \\ \gamma & I_{m_W} \end{pmatrix} \right].
\]

Substituting this in (8), pre-multiplying by $\left| \begin{pmatrix} 1 & 0 \\ -\gamma & I_{m_W} \end{pmatrix} \right|$, and post-multiplying by $\left| \begin{pmatrix} 1 & 0 \\ -\gamma & I_{m_W} \end{pmatrix} \right|$ yields

\[
\left| \kappa \hat{\Sigma} - (e \vdash Z \Pi_W + V_W) P_Z (e \vdash Z \Pi_W + V_W) \right| = 0.
\]

By a WLLN, we have $\Sigma^{1/2} \hat{\Sigma}^{-1/2} \rightarrow_p I_{1+m_W}$, and by (12), $\Sigma$ is positive definite. The matrix $\Sigma^{-1/2}$ therefore exists wpa1. Note that

\[
\Sigma^{-1/2} = \begin{pmatrix} \sigma_{ee}^{-1/2} & -\sigma_{ee}^{-1} \sigma_{eW} \Sigma_{WW,e}^{-1/2} \\ 0 & \Sigma_{WW,e}^{-1/2} \end{pmatrix} \quad \text{and} \quad \hat{\Sigma}^{-1/2} = \begin{pmatrix} \hat{\sigma}_{ee}^{-1/2} & -\hat{\sigma}_{ee}^{-1} \hat{\sigma}_{eW} \hat{\Sigma}_{WW,e}^{-1/2} \\ 0 & \hat{\Sigma}_{WW,e}^{-1/2} \end{pmatrix}.
\]
We pre- and post-multiply (33) by $|\Sigma^{-1/2}|$ and $|\Sigma^{-1/2}|$, respectively, to get

$$
|\kappa \Sigma^{-1/2} \hat{\Sigma}^{-1/2} - \Sigma_{W}^{-1/2} (e^\top Z \Pi W + V_W) P_Z (e^\top Z \Pi W + V_W) \Sigma_{W}^{-1/2} | = 0
$$

or

$$
|\kappa \Sigma^{-1/2} \hat{\Sigma}^{-1/2} - (z_{\epsilon,n}^\top \Theta_n + z_{\Pi W,n}) (z_{\epsilon,n}^\top \Theta_n + z_{\Pi W,n}) | = 0.
$$

We now use that

$$
(z_{\epsilon,n}^\top \Theta_n + z_{\Pi W,n}) (z_{\epsilon,n}^\top \Theta_n + z_{\Pi W,n})
$$

$$
= \left( z_{\epsilon,n} z_{\epsilon,n}^\top (\Theta_n + z_{\Pi W,n}) (\Theta_n + z_{\Pi W,n}) \right)
$$

$$
= N_n^\top L_n N_n
$$

to pre- and post-multiply the elements in the characteristic polynomial in (36) by $|\Sigma^{1/2} \hat{\Sigma}^{-1/2} |$ and $|\Sigma^{1/2} \hat{\Sigma}^{-1/2} |$, which exist wpa1:

$$
|\kappa I_{m_W+1} - (\Sigma^{1/2} \hat{\Sigma}^{-1/2})^\top N_n^\top L_n N_n (\Sigma^{1/2} \hat{\Sigma}^{-1/2}) | = 0.
$$

The smallest root $\kappa_{\min}$ of the characteristic polynomial in (38) is, with probability 1, equal to

$$
\min_{d \in [0,1]} \frac{d (\Sigma^{1/2} \hat{\Sigma}^{-1/2})^\top N_n^\top L_n N_n (\Sigma^{1/2} \hat{\Sigma}^{-1/2}) d}{d^\top d},
$$

which proves (21). If we now use a value of $d$ such that

$$
d = (\hat{\Sigma}^{1/2} \hat{\Sigma}^{-1/2}) \left( -\tau_n^\top \tau_n \right)^{-1/2} \eta_n
$$

the bottom $m_W$ rows of $N_n$ cancel out in the numerator and we obtain the bound

$$
AR_n(\beta_0) \leq \frac{1}{\xi_n} z_{\epsilon,n}^\top M_n z_{\epsilon,n}
$$
on the subset AR statistic. Q.E.D.

**Proof of Lemma 4:** For ease of presentation, we assume $\omega_n = n$. Using the moment restrictions in (12), an application of Lyapunov CLTs and WLLNs implies that, under any drifting parameter sequence $\lambda_n = (\gamma_n, \Pi_W, \Pi_Y, F_n)$,

$$
(z_{\epsilon,n}^\top, \text{vec}(z_{\Pi W,n}^\top)) \to_d (z_{\epsilon}^\top, \text{vec}(z_{\Pi W}^\top)) \sim N(0, I_{k(1+m_W)}),
$$

$$
Q^{-1}(n^{-1} Z^\top Z) \to_p I_k.
$$

Therefore, $z_{\epsilon,n}$ and $z_{\Pi W,n}$ are asymptotically independent.
Assume, without loss of generality, that the $j$th diagonal element $D_j$ of $D$ is finite for $j \leq p$ and $D_j = \infty$ for $j > p$, for some $0 \leq p \leq m_W$. Define a full rank diagonal matrix $B_n \in \Re^{m_W \times m_W}$ with $j$th diagonal element equal to 1 for $j \leq p$ and equal to $D_{nj}^{-1}$ otherwise for $j > p$. Note that, for all large enough $n$, the elements of $B_n$ are bounded by 1.

(i) We can write

\[ \Theta_n = (n^{-1} Z'Z)^{1/2} Q^{-1/2} \theta(n) = (n^{-1} Z'Z)^{1/2} G_n D_n R_n'. \]

Then, noting that $(n^{-1} Z'Z)^{1/2} Q^{-1/2} \rightarrow_p I_k$ under $\lambda_n$, we have $\Theta_n R_n B_n \rightarrow_p G \overline{D}$, where $\overline{D} \in \Re^{k \times m_W}$ is a rectangular diagonal matrix with diagonal elements $\overline{D}_j = D_j < \infty$ for $j \leq p$ and $\overline{D}_j = 1$ for $j > p$. Noting that $\Sigma^{-1/2} \hat{\Sigma}^{1/2} = I_{1+m_W} + o_p(1)$, we have

\[ \xi_n = (1, -\eta_n'(\tau_n'\tau_n)^{-1/2})(\Sigma^{-1/2} \hat{\Sigma}^{1/2})(1, -\eta_n'(\tau_n'\tau_n)^{-1/2})' + 1 + \eta_n'(\tau_n'\tau_n)^{-1} \eta_n + (1, e_n) + o_p(1), \]

for

\[ e_n = -z'_{e,n}(\tau_n R_n B_n)((\tau_n R_n B_n)'(\tau_n R_n B_n))^{-1}(R_n B_n)' \]

Note that $\tau_n R_n B_n = \Theta_n R_n B_n + z_{\overline{v}_W,n} R_n B_n$, $\Theta_n R_n B_n \rightarrow_p G \overline{D}$. Using (41) and $D_{nj}^{-1} \rightarrow 0$ for $j > p$, we have

\[ z_{\overline{v}_W,n} R_n B_n \rightarrow_d \tilde{z}_{\overline{v}_W} \equiv (z_{\overline{v}_W} R_1, \ldots, z_{\overline{v}_W} R_p, 0, \ldots, 0), \]

where $R_j$ denotes the $j$th column of $R$. We have $\text{vec}(z_{\overline{v}_W} R_1, \ldots, z_{\overline{v}_W} R_p) \sim N(0, I_{kp})$ because the columns of $R$ are orthogonal to each other. Therefore, $G \overline{D} + \tilde{z}_{\overline{v}_W}$ has full column rank with probability 1. This implies that

\[ ((\tau_n R_n B_n)'(\tau_n R_n B_n))^{-1} = O_p(1), \]

and given that $R_n B_n = O_p(1)$, we have $e_n = O_p(1)$. This and (43) then prove the claim with $p_n = \eta_n'(\tau_n'\tau_n)^{-1} \eta_n$.

(ii) Note that because $R_n B_n \in \Re^{m_W \times m_W}$ has full rank, we have $M_{\tau_n} = M_{\tau_n R_n B_n}$. As established in (i), we have $\tau_n R_n B_n \rightarrow_d G \overline{D} + \tilde{z}_{\overline{v}_W}$, where by (41), this limit is independent of the limit distribution $z_{\varepsilon} \sim N(0, I_k)$ of $z_{e,n}$. Therefore, $z'_{e,n} M_{\tau_n} z_{e,n} \rightarrow_d z'_{\varepsilon} M_{G \overline{D} + \tilde{z}_{\overline{v}_W}} z_{\varepsilon}$ under $\lambda_n$. Given independence of $z_{\varepsilon}$ and $\tilde{z}_{\overline{v}_W}$, it follows that, conditional on $\tilde{z}_{\overline{v}_W}$, we have $z'_{\varepsilon} M_{G \overline{D} + \tilde{z}_{\overline{v}_W}} z_{\varepsilon} \sim \chi^2_{k-m_W}$ whenever $G \overline{D} + \tilde{z}_{\overline{v}_W}$ has full column rank. Therefore, also unconditionally, $z'_{\varepsilon} M_{G \overline{D} + \tilde{z}_{\overline{v}_W}} z_{\varepsilon} \sim \chi^2_{k-m_W}$.

**Limiting Distribution of the Subset LM Statistic**

We next derive the limiting distribution of the subset LM statistic under the drifting sequence $\lambda_{n,h}$ in (20) in the weak IV case, where $\|h_{11}\| < \infty$.
and $\| h_{12} \| < \infty$. Recall that by WLLNs and CLTs, we have, under $\lambda_n$ for $\hat{Q} = n^{-1/2} Z'Z$,

$$(46) \begin{pmatrix} \hat{Q}^{-1/2} n^{-1/2} Z' \varepsilon / \sqrt{\sigma_{ee}} \\ \hat{Q}^{-1/2} n^{-1/2} Z' V'_Y / \sqrt{\sigma_{YY}} \\ \hat{Q}^{-1/2} n^{-1/2} Z' V'_W / \sqrt{\sigma_{WW}} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} z_{\varepsilon,h} \\ z_{V'_Y,h} \\ z_{V'_W,h} \end{pmatrix} \sim N \left( 0, \begin{pmatrix} 1 & h_{21} & h_{22} \\ h_{21} & 1 & h_{23} \\ h_{22} & h_{23} & 1 \end{pmatrix} \otimes I_k \right),$$

$$n^{-1} \begin{pmatrix} \varepsilon' \varepsilon & \sigma_{Y'}Y' & \sigma_{Y'}W' \\ \sigma_{ee} & \sigma_{YY} & \sigma_{YW} \\ \sigma_{ee} & \sigma_{YY} & \sigma_{WW} \end{pmatrix} \xrightarrow{p} (1, 1, 1, h_{21}, h_{22}, h_{23}),$$

$$Q^{-1} \hat{Q} \xrightarrow{p} I_k, \quad n^{-1} Z'[\varepsilon:V] \xrightarrow{p} 0,$$

where $z_{\varepsilon,h}$, $z_{V'_Y,h}$, $z_{V'_W,h} \in \mathbb{R}^k$. Define

$$(47) \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix} = \begin{pmatrix} (z_{V'_W,h} + h_{12})' (z_{V'_W,h} + h_{12}) \\ (z_{V'_W,h} + h_{12})' z_{\varepsilon,h} \end{pmatrix}.$$

It is easily shown that $(v_{1,h}, v_{2,h})'$ only depends on $h_{12}$ and not on the other elements in $h$. By Theorem 1(a) and Theorem 2 in Staiger and Stock (1997), we have

$$(48) \left( \frac{\sigma_{WW}}{\sigma_{ee}} \right)^{1/2} (\hat{\gamma} - \gamma) \xrightarrow{d} \Delta_h = \frac{v_{2,h} - \kappa_h h_{22}}{v_{1,h} - \kappa_h},$$

where $\kappa_h$ is the smallest root of the characteristic polynomial

$$(49) \left| (z_{\varepsilon,h}, z_{V'_W,h} + h_{12})' (z_{\varepsilon,h}, z_{V'_W,h} + h_{12}) - \kappa \Sigma_h \right| = 0$$

in $\kappa$ and $\Sigma_h \in \mathbb{R}^{2 \times 2}$ with diagonal elements 1 and off diagonal elements $h_{22}$. By Theorem 1(b) in Staiger and Stock (1997), we have

$$(50) \hat{\sigma}_{ee}(\beta_0, \hat{\gamma})/\sigma_{ee} \xrightarrow{d} \sigma_{ee}^2 = 1 - 2h_{22}\Delta_h + \Delta_h^2.$$

We have, from (46),

$$(51) \hat{Q}^{-1/2} n^{-1/2} Z'Y / \sqrt{\sigma_{YY}} \xrightarrow{d} z_{V'_Y,h} + h_{11},$$

$$\hat{Q}^{-1/2} n^{-1/2} Z'W / \sqrt{\sigma_{WW}} \xrightarrow{d} z_{V'_W,h} + h_{12}.$$

Note that it does not change the asymptotic results if one defines $\hat{\sigma}_{ee}(\beta_0, \hat{\gamma})$ with $M_Z$ replaced by $I_n$ as in Staiger and Stock (1997).
Combining (48)–(51), we obtain

\[
\hat{s} = (n^{-1}Z'Z)^{-1/2}n^{-1/2}Z'(y - Y\beta_0 - W\tilde{\gamma})/\sqrt{\sigma_{ee}} \rightarrow_d s_h
\]

\[
= -(z_{VW,h} + h_{12})\Delta_h + z_{e,h}.
\]

By (46), we have

\[
\hat{\sigma}_{eY}/(\sqrt{\sigma_{ee}\sigma_{YY}}) = (n-k)^{-1}(y - Y\beta_0 - W\tilde{\gamma})'M_ZY/(\sqrt{\sigma_{ee}\sigma_{YY}})
\]

\[
= (n-k)^{-1}(W(\gamma - \tilde{\gamma}) + \varepsilon)'M_ZY/(\sqrt{\sigma_{ee}\sigma_{YY}})
\]

\[
= (n-k)^{-1}(V_W(\gamma - \tilde{\gamma}) + \varepsilon)'M_ZY/(\sqrt{\sigma_{ee}\sigma_{YY}})
\]

\[
= \left(\frac{\sigma_{WW}}{\sigma_{ee}}\right)^{1/2}(\gamma - \tilde{\gamma})(n-k)^{-1}\frac{V_W'V_Y}{\sqrt{\sigma_{WW}\sigma_{YY}}}
\]

\[+ (n-k)^{-1}\frac{\varepsilon'V_Y}{\sqrt{\sigma_{ee}\sigma_{YY}}} + o_p(1),
\]

and likewise,

\[
\hat{\sigma}_{eW}/(\sqrt{\sigma_{ee}\sigma_{WW}}) = (n-k)^{-1}(y - Y\beta_0 - W\tilde{\gamma})'M_ZW/(\sqrt{\sigma_{ee}\sigma_{WW}})
\]

\[
= \left(\frac{\sigma_{WW}}{\sigma_{ee}}\right)^{1/2}(\gamma - \tilde{\gamma})(n-k)^{-1}\frac{V_W'V_W}{\sigma_{WW}}
\]

\[+ (n-k)^{-1}\frac{\varepsilon'V_W}{\sqrt{\sigma_{ee}\sigma_{WW}}} + o_p(1),
\]

where \(\hat{\sigma}_{eY}\) and \(\hat{\sigma}_{eW}\) have been implicitly defined here. Therefore, by (46) and (48),

\[
\hat{\sigma}_{eY}/(\sqrt{\sigma_{ee}\sigma_{YY}}) \rightarrow_d -\Delta_h h_{23} + h_{21} \quad \text{and}
\]

\[
\hat{\sigma}_{eW}/(\sqrt{\sigma_{ee}\sigma_{WW}}) \rightarrow_d -\Delta_h + h_{22}.
\]

Next, let \(\tilde{\Pi}(\beta_0) = (\tilde{\pi}_Y : \tilde{\pi}_W)\), \(\hat{\pi}_Y = (Z'Z)^{1/2}\tilde{\pi}_Y/\sqrt{\sigma_{YY}} \in \mathbb{R}^k\), and \(\hat{\pi}_W = (Z'Z)^{1/2}\tilde{\pi}_W/\sqrt{\sigma_{WW}} \in \mathbb{R}^k\). That is,

\[
\hat{\pi}_Y = \hat{Q}^{-1/2}n^{-1/2}Z\left[Y - (y - Y\beta_0 - W\tilde{\gamma})\hat{\sigma}_{eY}/\hat{\sigma}_{ee}(\beta_0, \tilde{\gamma})\right]/\sqrt{\sigma_{YY}}
\]

\[= \hat{Q}^{-1/2}n^{-1/2}Z'Y/\sqrt{\sigma_{YY}} - \hat{s}\hat{\sigma}_{eY}/(\sqrt{\sigma_{ee}\sigma_{YY}}) \hat{\sigma}_{ee}(\beta_0, \tilde{\gamma})/\sigma_{ee} \in \mathbb{R}^k,
\]

\[
\hat{\pi}_W = \hat{Q}^{-1/2}n^{-1/2}Z'W/\sqrt{\sigma_{WW}} - \hat{s}\hat{\sigma}_{eW}/(\sqrt{\sigma_{ee}\sigma_{WW}}) \hat{\sigma}_{ee}(\beta_0, \tilde{\gamma})/\sigma_{ee} \in \mathbb{R}^k.
\]
Using (50), (51), (52), and (55), we have

\[ \hat{p}_Y \rightarrow_d p_{Y,h} = z_{V_Y,h} + h_{11} - s_h \frac{-\Delta_h h_{23} + h_{21}}{\sigma_{eh}^2}, \]

\[ \hat{p}_W \rightarrow_d p_{W,h} = z_{V_W,h} + h_{12} - s_h \frac{-\Delta_h h_{22}}{\sigma_{eh}^2}. \]

By simple calculations,\(^9\)

\[ \text{LM}_n(\beta_0) = \left( \frac{\hat{s}_{ee}(\beta_0, \hat{\gamma})}{\sigma_{ee}} \right)^{-1} \hat{s}_{P(\hat{p}_Y, \hat{p}_W)} \hat{s}, \]

and therefore, by the continuous mapping theorem,

\[ \text{LM}_n(\beta_0) \rightarrow_d \text{LM}_h = s_h P(\hat{p}_{Y,h}, \hat{p}_{W,h}) s_h / \sigma_{eh}^2. \]

REFERENCES


\(^9\)Note that the numerical value of $\text{LM}_n(\beta_0)$ is not affected if one replaces $\hat{\Pi}(\beta_0)$ by $\hat{\Pi}(\beta_0)T$ for any invertible matrix $T \in \mathbb{R}^{2 \times 2}$. Here we take $T$ as a diagonal matrix with diagonal elements $\sigma_{YY}^{-1/2}, \sigma_{WW}^{-1/2}$.


Dept. of Economics, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA 92039-0508, U.S.A.; pguggenberger@ucsd.edu,

Dept. of Economics, Box B, Brown University, Providence, RI 02912, U.S.A.; Frank_Kleibergen@brown.edu,

Dept. of Economics, Oxford University, Manor Road, Oxford OX1 3UQ, United Kingdom; sophocles.mavroeidis@economics.ox.ac.uk,

and

Dept. of Economics, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA 92093-0534, U.S.A.; lic026@ucsd.edu.

Manuscript received November, 2009; final revision received July, 2012.